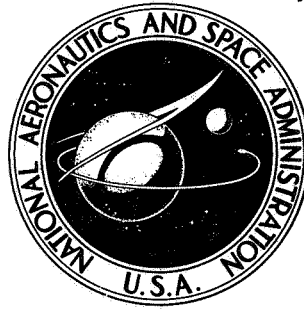


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# BOUNDARY-INTEGRAL METHODS IN ELASTICITY AND PLASTICITY

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# BOUNDARY-INTEGRAL METHODS IN ELASTICITY AND PLASTICITY

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## SUMMARY

Recently developed methods that use boundary-integral equations applied to elastic and elastoplastic boundary value problems are reviewed. Direct, indirect, and semi-direct methods using potential functions, stress functions, and displacement functions are described. Examples of the use of these methods for torsion problems, plane problems, and three-dimensional problems are given. It is concluded that the boundary-integral methods represent a powerful tool for the solution of elastic and elastoplastic problems.

## INTRODUCTION

Methods of analysis in solid mechanics, as in most other scientific and engineering fields, have been revolutionized by the advent of the modern digital computer. Thus, purely numerical methods, such as finite differences, and analytical methods, whose final stages require numerical procedures such as complex variable methods, awaited the modern computer for their practical implementation to all but the simplest problems.

Although many general methods have been and are being used for solving problems in elasticity and plasticity, the currently most popular ones probably are (1) finite differences, (2) finite elements, and (3) complex variables. To this list we may now add the boundary-integral methods. It is the boundary-integral methods that are discussed in this report.

First, the question might be asked, why is it necessary to get involved in yet another method for solving the same types of problems? What advantages do these methods have over the commonly used methods previously listed? The apparent advantages are as follows.

- (1) The need for conformal mapping is obviated.
- (2) Uniform or mixed boundary value problems are handled with equal ease.

(3) Stresses and displacements are obtained directly without need for numerical differentiation.

(4) No special treatment is needed for multiply connected regions.

(5) The internal stresses and displacements are obtained only where and when needed.

(6) The method can be extended directly to three-dimensional problems.

(7) Nodal points are needed only on the boundary instead of throughout the interior (as required by finite-difference or finite-element methods).

The last point, which is probably the most important one, is illustrated in figure 1. For the finite-difference or finite-element methods, the whole region must be covered by a grid, producing a large number of nodal points and corresponding unknowns. Thus a large number of simultaneous equations must be solved. In the present methods, as will shortly be seen, nodal points are taken only on the boundary, resulting in a much smaller number of unknowns, the dimensions of the problem being effectively reduced by one.

On the other hand, the resulting matrices are full, whereas in the finite-element methods, for example, the matrices are usually sparse and can be more or less banded.

The boundary integral methods themselves can be classified into three groups: indirect, semidirect, and direct.

The indirect methods formulate the problem in terms of unknown density functions, which have no physical significance. But once these density functions are determined, the displacements and/or stresses can be directly computed.

The semidirect methods formulate the problem in terms of unknown functions that are more familiar, such as the various stress functions. The stresses are then determined by simple differentiation.

The direct methods formulate the problem in terms of the direct physical quantities such as the displacements.

All three methods are illustrated in the next section by application to the Saint-Venant torsion problem. Also shown is how the solution can be extended to elastoplastic torsion. The more complicated plane strain and plane stress problems are then considered. We conclude with a brief discussion of three-dimensional problems.

The subject is approached herein from the applied viewpoint and not from the mathematical one. Such questions as the existence and uniqueness of solutions, the differences between internal and external problems, problems of convergence, etc., are not discussed. Excellent discussions of many of these questions can be found in references 1 to 6.

## TORSION PROBLEM

The torsion problem can be formulated in many ways (refs. 7 and 8):

(1) Warping function,  $W$ , where

$$\begin{aligned}\nabla^2 W &= 0 \quad \text{in } R \\ \frac{\partial W}{\partial n} &= (ly - mx)\alpha \quad \text{on } C\end{aligned}\tag{1}$$

where  $\alpha$  is the angle of twist per unit length and  $l$  and  $m$  are the direction cosines of the outward normal to the boundary  $C$  of the region  $R$ .

(2) Conjugate of warping function,  $V$ , where

$$\begin{aligned}\nabla^2 V &= 0 \quad \text{in } R \\ V &= \frac{G\alpha}{2} (x^2 + y^2) \quad \text{on } C\end{aligned}\tag{2}$$

with  $G$  equal to the shear modulus.

(3) Stress function,  $F$ , where

$$\begin{aligned}\nabla^2 F &= -2G\alpha \quad \text{in } R \\ F &= 0 \quad \text{on } C\end{aligned}\tag{3}$$

All three formulations represent classical problems in potential theory. This is why the methods to be discussed are frequently referred to as potential methods. Number (1) is the classical Neumann problem for Laplace's equation. Number (2) is the Dirichlet problem for Laplace's equation. And number (3) is the Dirichlet problem for Poisson's equation. In what follows, these three equations are referred to as formulations (1), (2), and (3).

### Formulation (1)

As is well known, the solution to Laplace's equation can be given in terms of either a single or double layer potential (refs. 1 and 2). For the warping function we take a single-layer potential, that is,

$$W(P) = \int_C \sigma(q) \log r_{Pq} dq\tag{4}$$

where (as shown in fig. 2)  $P$  represents an interior point,  $p$  and  $q$  are boundary points, and  $\sigma(q)$  is an as yet unknown boundary density function. Interior points are always designated by capital letters, and boundary points by lower case letters. Once  $\sigma(q)$  is determined, the warping function can be immediately calculated at any point. Incidentally, the point  $P$  can also lie on the boundary.

In order to determine the density distribution  $\sigma(q)$ , use is made of the boundary condition. Differentiating equation (4) and substituting into the second of equations (1), gives

$$W'(p) = \int_c \sigma(q) \left( \log r_{pq} \right)'_p dq - \pi \sigma(p) = (ly - mx)_p \alpha \quad (5)$$

where primes have been used to indicate normal derivatives and (as indicated in fig. 2) the normal derivative is taken positive outward. The subscript indicates the point at which the normal derivative is calculated. The term  $\pi \sigma(p)$  appearing in equation (5) arises from the fact that, although  $W(P)$  is continuous throughout the whole plane, its derivative suffers a jump at the boundary (ref. 1).

Equation (5) is an integral equation (Fredholm equation of the second kind) for the unknown boundary density function  $\sigma(q)$ . This is an example of an indirect potential method since the solution is given in terms of a density function that has no physical significance for the problem.

In order to solve equation (5) for the function  $\sigma(q)$ , the integral is replaced by a summation. The boundary  $c$  is divided into  $m$  intervals with a nodal point taken at the center of each interval. The function  $\sigma$  is assumed constant over each interval. Equation (5) then becomes

$$\sum_{i=1}^n (A_{ij} - \delta_{ij} \pi) \sigma_i = B_j \quad j = 1, n \quad (6)$$

where

$$A_{ij} = \int_i \left( \log r_{ij} \right)'_j dq$$

$$B_j = (ly - mx)_j \alpha$$

and where  $r_{ij}$  is the distance from the midpoint of the  $j^{\text{th}}$  interval to a point in the  $i^{\text{th}}$  interval and the derivative is evaluated at the midpoint of the  $j^{\text{th}}$  interval. The integration is taken over the  $i^{\text{th}}$  interval.

The set of  $n$  equations is solved for  $\sigma_i$ . The warping function  $W$  can then be computed at any point by equation (4). We see that there are only  $n$  equations to solve for the  $n$  boundary values of  $\sigma$ , rather than for a grid of nodal points throughout the region. The integrals over each interval can be evaluated in closed form (for straight boundaries), or a Simpson's rule or even trapezoidal rule may be used.

### Formulation (2)

Consider now the second formulation. Here we have two alternatives. Either a single- or double-layer potential may be used. Since the single-layer potential is continuous including the boundary, we have

$$V(p) = \int_c \sigma(q) \log r_{pq} dq = \frac{G\alpha}{2} (x_p^2 + y_p^2) \quad (7)$$

This is a singular Fredholm equation of the first kind for the unknown function  $\sigma(q)$ . Again, the integral is converted to a sum and solved for the  $\sigma_i$  at the nodal points.

$$\sum_{i=1}^n A_{ij} \sigma_i = B_j \quad (8)$$

where

$$A_{ij} = \int_i \log r_{ij} dq$$

$$B_j = \frac{G\alpha}{2} (x_j^2 + y_j^2)$$

Alternately, we can use a double-layer potential and write

$$V(P) = \int_c \mu(q) \left( \log r_{Pq} \right)'_q dq \quad (9)$$

Since the double-layer potential suffers a jump on reaching the boundary (refs. 1 and 2), we get

$$V(p) = \int_c \mu(q) \left( \log r_{pq} \right)'_q dq - \pi \mu(p) = \frac{G\alpha}{2} \left( x_p^2 + y_p^2 \right) \quad (10)$$

which is a singular Fredholm equation of the second kind for the unknown density function  $\mu(q)$ . Once equation (10) is solved for  $\mu(q)$ ,  $V(P)$  can be calculated everywhere from equation (9). As before, we replace equation (10) by

$$\sum_{i=1}^n (A_{ij} - \delta_{ij}\pi) \mu_i = B_j \quad (11)$$

where

$$A_{ij} = \int_i \left( \log r_{ij} \right)'_i dq$$

$$B_j = \frac{G\alpha}{2} \left( x_j^2 + y_j^2 \right)$$

Although at first sight it would seem that the formulation in terms of the Fredholm equation of the first kind is somewhat simpler, this equation has not been used often. The reason is that solutions of Fredholm's first equation are not as well understood as those of the second kind. In particular, the equation does not necessarily have a solution for every right hand side. The double-layer potential is thus usually used for the Dirichlet problem.

We now come to formulation (3). Here there is introduced a new approach which is more general than either the single-layer or double-layer potentials. Furthermore, the previous solutions were indirect in the sense that they involved solutions for physically meaningless density functions, but the new method is semidirect.

### Formulation (3)

Equation (3) is an inhomogeneous equation for the stress function  $F$ . Neither the single- or double-layer potentials will satisfy the inhomogeneous equation because a particular solution is needed. However, a solution can be obtained in a different form that satisfies both homogeneous and inhomogeneous equations by the use of Green's formula or, as it is sometimes called Green's third identity.



Given two functions  $u(x, y)$  and  $v(x, y)$  with continuous first and second derivatives in the region  $R$ , Green's second theorem states

$$\iint_R (u \nabla^2 v - v \nabla^2 u) dx dy = \int_c \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dq \quad (12)$$

If we let  $u = F$  and  $v = \log r$ , this equation results in (see the appendix)

$$2\pi F(P) = \iint_R f(Q) \log r_{PQ} dx dy + \int_c \left[ F(\log r_{Pq})'_q - F' \log r_{Pq} \right] dq \quad (13)$$

where for our problem  $f(Q) = -2G\alpha$ .

This equation gives the values of  $F$  at any interior point  $p$  in terms of  $F$  and  $F'$  on the boundary. We notice that the contour integral represents both a single-layer and a double-layer potential, but, instead of being artificial density functions, they are proportional to the distributions of the stress function and its derivative on the boundary. We might therefore designate this type of solution as semidirect.

Since the double layer potential jumps as we approach the boundary, equation (13) becomes for  $P$  on the boundary

$$\pi F(p) = \iint_R f(Q) \log r_{pQ} dx dy + \int_c \left[ F(\log r_{pq})'_q - F' \log r_{pq} \right] dq \quad (14)$$

This solution has many advantages.

(1) It works for an inhomogeneous equation. This is particularly important for the elastoplastic problem as will shortly be seen.

(2) It takes care of any boundary condition. If  $F$  is prescribed (Dirichlet problem), equation (14) becomes an integral equation for  $F'$  on the boundary. If  $F'$  is prescribed (Neumann problem), it becomes an integral equation for the unknown  $F$  on the boundary. If  $F$  is prescribed over part of the boundary and  $F'$  on the other part (mixed boundary value or Hilbert problem), then equation (14) is an integral equation for those parts of  $F$  and  $F'$  that are unknown.

(3) The unknown functions are physically more meaningful.

The solution of equation (14) can be obtained exactly as previously described by replacing the integrals by summations.

To calculate the stresses from the stress function, it is not necessary to carry out any numerical differentiation of the stress function. We can differentiate under the integral sign in equation (13). Then we can calculate the stresses by the same type of numer-

ical integration as before, once  $F$  and  $F'$  are known on the boundary. This is a more accurate way of computing the stresses.

The use of Green's formula is of course not restricted to formulation (3) for the stress function. It can just as well be used for determining  $W$  or  $V$ . We just replace  $F$  by  $W$  in equations (13) and (14) and set  $f$  equal to zero. This solution is then referred to as a direct potential method, since we solve directly for the physical quantity desired, the warping function.

### Some Numerical Results

A number of torsion problems have been solved by Jaswon and Ponter (ref. 9), including prismatic bars with cross sections consisting of solid and hollow ellipses, rectangles, equilateral triangles, and circles with curved notches. The direct potential method was used to determine the warping function. A few of the results taken from this reference are shown in table I. Results obtained by the present author for the square cross section using the stress function rather than the warping function were essentially the same. Excellent agreement is obtained with the known analytical results.

### Elastoplastic Torsion

By use of Green's boundary formula we can treat the elastoplastic problem the same way as the elastic problem. With plastic flow occurring, the equation for the stress function can be written as

$$\nabla^2 F = -2G\alpha - 2G \left( \frac{\partial \epsilon_{zx}^p}{\partial y} - \frac{\partial \epsilon_{zy}^p}{\partial x} \right) \equiv f(x, y) \quad (15)$$

The function  $f$  in the double integral of Green's formula (eq. (14)) now contains the plastic flow terms indicated. These terms are computed by the use of the usual plasticity theory. The solution is an iterative one.

We start by assuming the plastic strains to be zero. The problem is then solved using Green's formula as indicated previously, and the stresses are computed by differentiation. One then computes the total strains from the relations

and

$$\left. \begin{aligned} \epsilon_{zx} &= \frac{1}{2G} \sigma_{zx} + \epsilon_{zx}^p \\ \epsilon_{zy} &= \frac{1}{2G} \sigma_{zy} + \epsilon_{zy}^p \end{aligned} \right\} \quad (16)$$

and the equivalent total strains from

$$\epsilon_{et} = \frac{2}{\sqrt{3}} \sqrt{\epsilon_{zx}^2 + \epsilon_{zy}^2} \quad (17)$$

From the stress-strain curve of the material the equivalent, plastic strain is then obtained:

$$\epsilon_p = f(\epsilon_{et}) \quad (18)$$

And the plastic strains are then given by

and

$$\left. \begin{aligned} \epsilon_{zx}^p &= \frac{\epsilon_p}{\epsilon_{et}} \epsilon_{zx} \\ \epsilon_{zy}^p &= \frac{\epsilon_p}{\epsilon_{et}} \epsilon_{zy} \end{aligned} \right\} \quad (19)$$

First approximations to the plastic strains and consequently for the function  $f(x,y)$  appearing in equation (15) have thus been determined. The process is then repeated until convergence is obtained. A more detailed discussion of the successive approximation method (or the "method of initial strains," as it is sometimes called) can be found in reference 10. Note that only the area integral, which represents the right hand side of the set of equations to be solved, changes from iteration to iteration. Note also that, although total plastic strains have been used in the previous equations for ease of writing, incremental strains could have been used just as well.

To illustrate the correctness of this type of formulation we can consider the problem of a circular shaft for which the solution can be obtained in closed form if linear strain-hardening of the material is assumed.

Consider a circular shaft of radius  $a$ . The radial coordinate will be designated by  $\rho$ , to distinguish it from  $r$ , the distance between the fixed point and the variable point appearing in the previous formulas. In polar coordinates, because of symmetry, the function  $f$  appearing in equation (15) becomes

$$f = -2G\alpha + \frac{2G}{\rho} \frac{\partial}{\partial \rho} \left( \rho \epsilon_{z\theta}^p \right) \quad (20)$$

For linear strain hardening, it follows that (ref. 10, p. 255)

$$\epsilon_{z\theta}^p = A\rho + B \quad (21)$$

where

$$A = \frac{3\alpha}{2 \left[ 3 + 2(1 + \nu) \frac{m}{(1 - m)} \right]}$$

$$B = \frac{-\sqrt{3} (1 + \nu) \epsilon_0}{3 + 2(1 + \nu) \frac{m}{(1 - m)}}$$

where  $\nu$  is Poisson's ratio,  $m$  is the strain hardening parameter (ratio of the slope of the strain hardening line to the elastic modulus), and  $\epsilon_0$  is the yield strain.

On the boundary  $F(a) = 0$  and, because of axial symmetry,  $F'(a) = \text{a constant}$ . Green's boundary formula (eq. (14)) then becomes

$$0 = 2G \iint \left( 2A - \alpha + \frac{B}{\rho} \log r_{pQ} \right) dx dy - F'(a) \int_c \log r_{pq} dq \quad (22)$$

which, upon solving for  $F'(a)$ , gives

$$F'(a) = G[a(2A - \alpha) + 2B]$$

Hence, for any interior point

$$2\pi F(\rho) = 2G \iint \left( 2A - \alpha + \frac{B}{\rho} \right) \log r_{\rho q} dx dy - F'(a) \int_c \log r_{\rho q} dq \quad (23)$$

or

$$F(\rho) = \frac{G}{2} \left[ (2A - \alpha)(\rho^2 - a^2) + 4B(\rho - a) \right]$$

and the shear stress  $\tau$  is given by

$$\tau = -\frac{\partial F}{\partial \rho} = 2G \left[ \left( \frac{\alpha}{2} - A \right) \rho - B \right] \quad (24)$$

which agrees with the solution obtained in an entirely different fashion in reference 10. Note that this solution is valid only in the plastic region, that is, for  $\rho \geq \rho_c$  where  $\rho_c$ , the elastic-plastic boundary, is given by (ref. 10),

$$\rho_c = \frac{2(1 + \nu)\epsilon_0}{\sqrt{3}\alpha} \quad (25)$$

For  $\rho \leq \rho_c$  the usual elastic solution prevails.

## THE PLANE PROBLEM

We now consider the plane problem of elastoplasticity, that is, plane strain or plane stress. The problem can be formulated in several ways: (1) by the use of the Airy stress function, which is a semidirect formulation, (2) by the use of singular solutions of the Navier equations, which is a direct formulation, or (3) by the use of fictitious loads, which is an indirect formulation. The second approach lends itself directly to the three-dimensional extension. The last method cannot easily take into account plastic flow.

### Formulation in Terms of Airy Stress Function

In terms of the Airy stress function, we have to solve the problem (for plane stress),

$$\nabla^4 F = f(x, y) \quad \text{in } R \quad (26)$$

where  $F$  and  $\partial F / \partial n$  are given on the boundary, and

$$f(x, y) = -E\nabla^2(\alpha T) - E \left( \frac{\partial^2 \epsilon_x^p}{\partial y^2} + \frac{\partial^2 \epsilon_y^p}{\partial x^2} - 2 \frac{\partial^2 \epsilon_{xy}^p}{\partial x \partial y} \right) \quad (27)$$

For plane strain this function is slightly different (ref. 10). For an elastic isothermal problem  $f(x, y) = 0$ . Let

$$\varphi \equiv \nabla^2 F \quad (28)$$

Then,

$$\nabla^2 \varphi = f(x, y) \quad (29)$$

We now have a Poisson equation for  $\varphi$  and an inhomogeneous biharmonic equation for  $F$ . Using the same Green's boundary formula as before for  $\varphi$  results in

$$\pi \varphi(p) = \iint_R f(\xi, \eta) \log r_{pQ} d\xi d\eta + \int_c \left[ \varphi (\log r_{pq})'_q - \varphi' \log r_{pq} \right] dq \quad (30)$$

For the biharmonic equation we can obtain a similar boundary formula using Green's second theorem (see the appendix):

$$4\pi F(p) = \iint_R \rho_{pQ} f(\xi, \eta) d\xi d\eta + \int_c \left[ F (\nabla^2 \rho_{pq})'_q - F' \nabla^2 \rho_{pq} + (\rho_{pq})'_q \varphi - \rho_{pq} \varphi' \right] dq \quad (31)$$

where

$$\rho \equiv r^2 \log r$$

We have thus two coupled integral equations for the two unknown functions  $\varphi$  and  $\varphi'$  on the boundary. These equations can be solved as before. The boundary is divided into  $m$  intervals, and  $F$ ,  $F'$ ,  $\varphi$ , and  $\varphi'$  are assumed constant over each interval. The integrals are replaced by summations, giving

$$\left. \begin{aligned} \pi \varphi_i &= \sum_{j=1}^m (a_{ij} \varphi_j + b_{ij} \varphi_j') + J_i \quad i = 1, 2, \dots, m \\ 4\pi F_i &= \sum_{j=1}^m (c_{ij} \varphi_j + d_{ij} \varphi_j' + e_{ij} F_j + f_{ij} F_j') + K_i \end{aligned} \right\} \quad (32)$$

where

$$a_{ij} = \int_j \frac{\partial \log r_{ij}}{\partial n} dq$$

$$b_{ij} = - \int_j \log r_{ij} dq$$

$$c_{ij} = \int_j \frac{\partial \rho_{ij}}{\partial n} dq$$

$$d_{ij} = - \int_j \rho_{ij} dq$$

$$e_{ij} = \int_j \frac{\partial \nabla^2 \rho_{ij}}{\partial n} dq$$

$$f_{ij} = - \int_j \nabla^2 \rho_{ij} dq$$

$$J_i = \iint_R f(Q) \log r_{iQ} d\xi d\eta$$

$$K_i = \iint_R f(Q) \rho_{iQ} d\xi d\eta$$

We thus have  $2m$  equations to solve for the  $2m$  unknowns. If part of the boundary happens to be a line of symmetry, then  $F'$  and  $\varphi'$  are zero there, and the number of unknowns is reduced. Once this solution is obtained, the stress function  $F$  at any interior point is obtained from (see the appendix)

$$8\pi F(P) = \iint_R \rho_{PQ} f(\xi, \eta) d\xi d\eta + \int_c \left[ F(\nabla^2 \rho_{PQ})'_q - F' \nabla^2 \rho_{PQ} + (\rho_{PQ})'_q \varphi - \rho_{PQ} \varphi' \right] dq \quad (33)$$

As a simple trivial example, consider an elastic square plate under uniform tension with unit load (as shown in fig. 3). By taking just four intervals (each side is one interval), we get

$$\sigma_x = 1.0000$$

We now consider a problem that is by no means trivial, namely, the elastic thermal stress in a square plate under a parabolic temperature distribution. The solution has been obtained in various ways, the most accurate probably being a finite difference solution using a 20 by 20 grid and thus involving the solution of 400 simultaneous equations. A comparison of that result (ref. 11) with the present method using 60 intervals on the boundary is shown in figure 4. Actually, because of double symmetry, there are only 15 equations to solve. As shown by the figure, good agreement was obtained.

A similar formulation has been used for plate bending problems in reference 12.

Let us now consider a problem for which no good solution has been available heretofore, namely, the elastoplastic problem of an edge-notched beam under pure bending (fig. 5).

The function  $f(x,y)$  appearing in equation (31) is the right hand side of equation (27) with  $T = 0$ . An iterative solution is performed as for the torsion problem except that the plane stress or plane strain plasticity equations are used (ref. 10).

Figure 5 shows the spread of the plastic zone with increasing load for the plane strain case. A complete discussion of this problem is presented in references 13 and 14.

#### Formulation in Terms of Potential Functions

The preceding problems can also be formulated in terms of single-layer density functions as for the torsion problem. This approach has been investigated in some detail in references 3 to 5 and 15 to 17. This formulation is based on the fact that a biharmonic function can always be represented in terms of two harmonic functions as shown in standard elasticity books. Thus,

$$F(P) = r_P^2 \alpha(P) + \beta(P) \quad (34)$$

where  $\alpha$  and  $\beta$  are harmonic functions and can therefore be represented in terms of single- or double-layer potentials and  $r_P$  is the distance from the point  $P$  to the origin of coordinates. We can thus write

$$\begin{aligned} \alpha(P) &= \int_C \mu(q) \log r_{Pq} dq \\ \beta(P) &= \int_C \sigma(q) \log r_{Pq} dq \end{aligned} \quad (35)$$

and substituting into equation (34) on the boundary gives



$$\left. \begin{aligned}
F(p) &= r_p^2 \int_c \mu(q) \log r_{pq} dq + \int_c \sigma(q) \log r_{pq} dq \\
F'(p) &= -\pi \left[ r_p^2 \mu(p) + \sigma(p) \right] + 2r_p' r_p \int_c \mu(q) \log r_{pq} dq \\
&\quad + r_p^2 \int_c \mu(q) (\log r_{pq})'_p dq + \int_c \sigma(q) (\log r_{pq})'_p dq
\end{aligned} \right\} \quad (36)$$

which results again in two coupled integral equations for the unknown density functions  $\mu$  and  $\sigma$ . These equations can be solved as previously indicated. A number of solutions using this approach have been obtained by Jaswon and by Rim as previously noted.

Alternately, we can use Green's boundary formula for the two potentials. Thus

$$\begin{aligned}
\pi \alpha(p) &= \int_c \alpha(q) (\log r_{pq})'_q dq - \int_c \alpha'(q) \log r_{pq} dq \\
\pi \beta(p) &= \int_c \beta(q) (\log r_{pq})'_q dq - \int_c \beta'(q) \log r_{pq} dq
\end{aligned} \quad (37)$$

With  $F$  and  $F'$  given on the boundary, we can write

$$\beta(q) = F(q) - r_q^2 \alpha(q) \quad (38)$$

$$\beta'(q) = F'(q) - [r_q^2 \alpha(q)]'$$

Eliminating  $\beta$  and  $\beta'$  from the second of equations (37) gives

$$\pi [F(p) - r_p^2 \alpha(p)] = \int_c [F(q) - r_q^2 \alpha(q)] (\log r_{pq})'_q dq - \int_c \{F'(q) - [r_q^2 \alpha(q)]'\} \log r_{pq} dq \quad (39)$$

We thus have, as before, a pair of coupled integral equations for the two unknown functions  $\alpha$  and  $\alpha'$ , which can be solved as previously.

A variation of this formulation that is advantageous is given in reference 18. Therein  $\alpha$  and  $\alpha'$  are eliminated from equation (39) by using the first of equations (35) and its normal derivative on the boundary. This gives an integral equation in just one unknown density function  $\mu(q)$ , namely,

$$\begin{aligned}
\pi \left[ F(p) - r_p^2 \int_c \mu(q) \log r_{pq} dq \right] &= \int_c \left[ F(q) - r_q^2 \int_c \mu(q_1) \log r_{qq_1} dq_1 \right] \left( \log r_{pq} \right)'_q dq \\
&- \int_c \left[ F'(q) - \left( r_q^2 \right)'_q \int_c \mu(q_1) \log r_{qq_1} dq_1 - r_q^2 \int_c \mu(q_1) \left( \log r_{qq_1} \right)'_q dq_1 \right. \\
&\left. + \pi r_q^2 \mu(q) \right] \log r_{pq} dq
\end{aligned} \tag{40}$$

Equation (40) contains only one unknown function:  $\mu(q)$ . It can be solved numerically (as previously indicated) by dividing the boundary into  $n$  segments and writing the integrals in summation form. This solution yields  $n$  simultaneous equations for the  $n$  unknown values of  $\mu$ . Once  $\mu$  is known, the function  $\alpha$  is calculated from the first of equations (35).

To determine the function  $\beta$ , its values on the boundary  $\beta(p)$  are first calculated from the first of equations (38). Then expressing  $\beta$  in terms of a double-layer potential (similar to eqs. (9) and (10)), we have

$$\begin{aligned}
\beta(P) &= \int_c \sigma(q) \left( \log r_{Pq} \right)'_q dq \\
\beta(p) &= \int_c \sigma(q) \left( \log r_{pq} \right)'_q dq - \pi \sigma(p)
\end{aligned} \tag{41}$$

where  $\sigma(p)$  is an unknown density function, which is now determined by solving the second of equations (41). This again requires a solution of  $n$  simultaneous equations as before. Once  $\sigma(p)$  is known,  $\beta(P)$  is determined from the first of equations (41). This completes the solution since  $F$  can now be computed everywhere from equation (34).

The advantage of this approach, even though it is somewhat more complicated, is that, instead of having to solve  $2n$  simultaneous equations in the case of  $n$  boundary intervals, we solve separately two sets of  $n$  simultaneous equations each. This is advantageous from the viewpoints of computer running time and storage, loss of significant figures, and round-off errors.

## Formulation in Terms of Fictitious Loads

A somewhat different form of indirect formulation was proposed by Massonnet (ref. 19). This method makes use of the superposition of fictitious loads on the boundary of the region in such a way that the boundary conditions are satisfied. These fictitious loads correspond to singular solutions of Laplace's equation and are therefore completely analogous to the potential of a double layer.

For these fictitious loads one can make use of the fundamental stress state called the "radial simple stress state" due to a concentrated force  $P$  acting on the boundary of the half space. If we consider a distribution of these forces  $P(q)$  acting on the boundary of the body, then the resultant stress vector acting on a plane through an interior point  $M$  (as shown in fig. 6(a)) will be given by (ref. 19)

$$\bar{T}_n = -\frac{2}{\pi} \int_C P(q) \frac{\cos \varphi \cos \alpha}{r} \bar{e}_r dq \quad (42)$$

where  $\varphi$  is the angle between  $\bar{P}$  and  $r$ ,  $\alpha$  is the angle between  $n$  and  $r$  and where  $\bar{e}_r$  is the unit vector in the direction of  $r$ . It should be emphasized that the loads  $P(q)$  have nothing in common with the actual boundary loads. They are purely fictitious loads.

If the point  $M$  moves to the point  $R$  on the boundary, the stress vector  $\bar{T}$  jumps discontinuously because we must add to  $\bar{T}$  the value of  $\bar{P}(R)$  itself acting at this point. Furthermore on the boundary the resultant force must equal the applied boundary load. Thus on the boundary we must have

$$\bar{T}(R) = \bar{P}(R) - \frac{2}{\pi} \int_C P(q) \frac{\cos \varphi \cos \alpha}{r} \bar{e}_r dq \quad (43)$$

where  $\bar{T}$  must equal the applied boundary load vector.

The problem thus reduces to finding a fictitious load distribution  $P(q)$ , or density function, satisfying equation (43). Once this function is determined, the stress vector at any interior point can be calculated from equation (42).

In rectangular coordinates equation (43) can be written as (see fig. 6(b))

$$\left. \begin{aligned} T_x &= P_x - \frac{2}{\pi} \int_C A(P) \beta_q R_x^r dq \\ T_y &= P_y - \frac{2}{\pi} \int_C A(P) \beta_q R_y^r dq \end{aligned} \right\} \quad (44)$$

where

$$A(P) \equiv P_x r_x + P_y r_y$$

is a function of the fictitious loads  $P$  and where

$$\beta_{qR} \equiv \frac{n_x r_x + n_y r_y}{r^4}$$

is a function only of the geometry. Again, the integrals can be replaced by summations.

A similar type of formulation was presented by Liu and coworkers (refs. 20 to 22), who also considered multiply connected regions, bodies with axial symmetry, and extended the formulation to three-dimensional problems as well. We note that the method is not directly applicable to bodies with displacement boundary conditions. A similar type of approach has also been used by Oliveira (ref. 23) using complex potentials.

### Direct Formulation

We now come to a direct formulation of the two-dimensional problem which is directly extendable to three-dimensional problems. This formulation is based on a singular solution of the Navier equations.

The Navier equations for plane strain, including plastic flow, temperature, and body force terms, are

$$\left. \begin{aligned} \nabla^2 u_i + \frac{1}{1-2\nu} \theta_{,i} &= 2\epsilon_{ij,j}^p + 2 \frac{1+\nu}{1-2\nu} (\alpha T)_{,i} - \frac{F_i}{G} \\ \theta &\equiv \epsilon_{jj} = u_{j,j} \quad i, j = 1, 2 \end{aligned} \right\} \quad (45)$$

Henceforth the usual tensor notation will be used, where a repeated subscript indicates summation over its range and a comma indicates partial differentiation. In equation (45)  $\epsilon_{ij}^p$  represents the sum of all the plastic strain increments up to and including the current increment of load,  $\nu$  is Poisson's ratio,  $\alpha$  is the coefficient of linear thermal expansion, and  $G$  is the shear modulus.

For plane stress the equation becomes

$$\nabla^2 u_i + \frac{1+\nu}{1-\nu} \theta_{,i} = 2\epsilon_{ij,j}^p + \frac{2\nu}{1-\nu} \theta_{,i}^p + \frac{2(1+\nu)}{1-\nu} (\alpha T)_{,i} - \frac{F_i}{G} \quad (45a)$$

where

$$\theta^p = \epsilon_x^p + \epsilon_y^p$$

The boundary conditions that must be satisfied by the displacements over that part of the boundary where forces  $P_i$  are specified are

$$\frac{P_i}{G} = (u_{i,j} + u_{j,i})n_j + \frac{2\nu}{1-2\nu} u_{k,k}n_i - 2 \left( \epsilon_{ij}^p n_j + \frac{1+\nu}{1-2\nu} \alpha T n_i \right) \quad (46)$$

for plane strain and

$$\frac{P_i}{G} = (u_{i,j} + u_{j,i})n_j + \frac{2\nu}{1-\nu} u_{k,k}n_i - 2 \left[ \epsilon_{ij}^p n_j - \left( \frac{\nu}{1-\nu} \theta^p - \frac{1+\nu}{1-\nu} \alpha T \right) n_i \right] \quad (46a)$$

for plane stress.

We now note the following interesting fact: By defining a pseudobody force by

$$F_i' = F_i - 2G \left[ \epsilon_{ij,j}^p + \frac{1+\nu}{1-2\nu} (\alpha T)_{,i} \right] \quad (47)$$

and a pseudoboundary force by

$$P_i' = P_i + 2G \left( \epsilon_{ij}^p n_j + \frac{1+\nu}{1-2\nu} \alpha T n_i \right) \quad (47a)$$

we may write equations (45) and (46) for plane strain as

$$\left. \begin{aligned} \nabla^2 u_i + \frac{1}{1-2\nu} \theta_{,i} &= -\frac{F_i'}{G} \\ \frac{P_i'}{G} &= (u_{i,j} + u_{j,i})n_j + \frac{2\nu}{1-2\nu} u_{k,k}n_i \end{aligned} \right\} \quad (48)$$

We thus have to solve a problem with modified body forces and modified boundary loads. This is a direct generalization of the Duhamel-Neumann analogy for thermoelastic problems (ref. 24). Equations (48) could have been obtained by assuming an elastic body with body forces given by equation (47) and surface forces given by (47a) (for plane strain). For this imaginary body the equilibrium equations would contain the plastic flow terms together with the body forces, if any, but the stress-strain relations would not. The displacements obtained by solving equations (48) would thus be correct. The stresses, however, would be pseudostresses. The true stresses have to be computed from the correct stress-strain relations. It should also be emphasized that the actual pseudobody and pseudoboundary forces are not known apriori and have to be determined as part of the solution by, for example, an iterative process. For plane stress,  $P'$  and  $F'$  become

$$\left. \begin{aligned} F'_i &= F_i - 2G \left[ \epsilon_{ij,j}^p + \frac{\nu}{1-\nu} \theta_{,i}^p + \frac{1+\nu}{1-\nu} (\alpha T)_{,i} \right] \\ P'_i &= P_i + 2G \left[ \epsilon_{ij}^p n_j + \left( \frac{\nu}{1-\nu} \theta^p - \frac{1+\nu}{1-\nu} \alpha T \right) n_i \right] \end{aligned} \right\} \quad (49)$$

A solution of the nonhomogeneous Navier equations can be obtained by making use of the singular solution of the Navier equations due to a point load, as given for example in Love's classical book on elasticity (ref. 25), and also making use of Betti's reciprocal theorem. We then arrive at a solution (see the appendix) that is known as Somigliana's identity, namely,

$$\lambda u_i(P) = \int_C (U_{ij} P'_j - T_{ij} u_j) dq + \int_R U_{ij} F'_j dA \quad (50)$$

where

$$\left. \begin{aligned} U_{ij} &= -\frac{1}{8\pi G(1-\nu)} \left[ \delta_{ij}(3-\nu) \log r - r_{,i} r_{,j} \right] \\ T_{ij} &= -\frac{1}{4\pi(1-\nu)r} \left\{ \left[ \delta_{ij}(1-2\nu) + 2r_{,i} r_{,j} \right] \frac{\partial r}{\partial n} - (1-2\nu)(r_{,i} n_j - r_{,j} n_i) \right\} \end{aligned} \right\} \quad (51)$$

and  $F'$  and  $P'$  are given by equations (47) and (47a). The coefficient  $\lambda$  is equal to 1, if  $P$  is an interior point and is equal to 1/2, if  $P = p$  is a boundary point. For plane stress,  $\nu$  in equation (51) is replaced by  $\nu/(1+\nu)$ , and  $P'$  and  $F'$  are taken from

equations (49). The derivatives appearing in equation (51) are taken at the variable point of integration.

Instead of using the pseudobody and pseudoboundary forces in equation (50), an alternate solution can be written as follows (see the appendix)

$$\lambda u_i(P) = \int_C (U_{ij} P_j - T_{ij} u_j) dq + \int_R U_{ij} F_j dA + \int_R \Sigma_{jki} (\epsilon_{jk}^p + \delta_{jk} \alpha T) dA \quad (52)$$

where

$$\Sigma_{jki} = -\frac{1}{4\pi(1-\nu)r} \left[ (\delta_{ij}^r{}_{,k} + \delta_{ik}^r{}_{,j} - \delta_{jk}^r{}_{,i})(1-2\nu) + 2r_{,i}^r{}_{,j}{}_{,k} \right] \quad (53)$$

It can be shown that equations (50) and (52) are identical. Formulation (52) has the advantage that it uses the physically given boundary conditions and, more importantly, that it does not require the derivatives of the plastic strains.

The solution is now obtained by replacing the integrals by sums as before. Note that the displacements and, hence, the stress fields are given directly in terms of the boundary forces and boundary displacements. If the displacements are given on the boundary, we have an integral equation for the boundary forces. If the forces are given on the boundary, we have an integral equation for the unknown boundary displacements. If the forces are given over part of the boundary and the displacements over the rest of the boundary (mixed boundary value problem), we have an integral equation for those parts of the boundary loads and displacements that are unknown. Also, since we are working directly with displacements, there is no difficulty with multiply connected regions.

The stresses at any interior point  $P$  are obtained by direct differentiation of equation (52), resulting in

$$\sigma_{ij}(P) = \int_C (V_{ijk} P_k - T_{ijk} u_k) dq + \int_R V_{ijk} F_k dA - 2G\epsilon_{ij}^p + \int_R T_{ijk} (\epsilon_{kl}^p + \delta_{kl} \alpha T) dA \quad (54)$$

where

$$\left. \begin{aligned} V_{ijk} &= -G \left( U_{ik,j} + U_{jk,i} + \delta_{ij} \frac{2\nu}{1-2\nu} U_{lk,l} \right) \\ T_{ijk} &= -G \left( T_{ik,j} + T_{jk,i} + \delta_{ij} \frac{2\nu}{1-2\nu} T_{lk,l} \right) \\ T_{ijk,l} &= -G \left( \Sigma_{kli,j} + \Sigma_{klj,i} + \delta_{ij} \frac{2\nu}{1-2\nu} \Sigma_{klm,m} \right) \end{aligned} \right\} \quad (55)$$

As previously noted all derivatives are taken at the variable point of integration. For the elastic problem all the plastic flow terms disappear.

Solution to a number of elastic problems using these equations have been obtained, for example, in references 6 and 26 to 30. These include problems of circular and elliptic regions, rings, rectangles, bodies with holes, inclusions, notches, and cracks as well as some thermoelastic problems.

The Navier equations (45) and (46) and the corresponding solution (52) have been written in terms of displacements, stresses, and plastic strains. This is the appropriate form if the method of successive elastic solutions, or method of initial strains (ref. 10), is used to solve the elastoplastic problem. If the tangent modulus method is used (refs. 31 to 33), these equations are written in terms of velocities, stress rates, and plastic strain rates. The boundary-integral formulation remains the same.

### THREE-DIMENSIONAL PROBLEMS

For three-dimensional problems the extension is direct. The Navier equations (45) retain the same form, except that the range of subscripts is now three and the Laplacian appearing in these equations is the three-dimensional Laplacian. The singular solution, instead of having a logarithmic singularity, has a  $1/r$  singularity. The solution then takes the same form as before, except that the kernels of the integrals are slightly different. The area integral becomes a volume integral, and the line integrals become surface integrals over the bounding surface of the body. Thus equation (52) becomes

$$\lambda u_i(P) = \int_S (U_{ij} P_j - T_{ij} u_j) dS + \int_V U_{ij} F_j dV + \int_V \Sigma_{jki} (\epsilon_{jk}^p + \delta_{jk} \alpha T) dV \quad (56)$$

where



$$\left. \begin{aligned} U_{ij} &+ \frac{1}{16\pi(1-\nu)G} \frac{1}{r} \left[ (3-4\nu)\delta_{ij} + r_{,i}r_{,j} \right] \\ T_{ij} &= -\frac{1}{8\pi(1-\nu)} \frac{1}{r^2} \left\{ \left[ (1-2\nu)\delta_{ij} + 3r_{,i}r_{,j} \right] \frac{\partial r}{\partial n} + (1-2\nu)(r_{,j}n_i - r_{,i}n_j) \right\} \\ \Sigma_{jki} &= -\frac{1}{8\pi(1-\nu)} \frac{1}{r^2} \left[ (1-2\nu)(\delta_{ij}r_{,k} + \delta_{ik}r_{,j} - \delta_{jk}r_{,i}) + 3r_{,i}r_{,j}r_{,k} \right] \end{aligned} \right\} \quad (57)$$

Instead of dividing the boundary into a set of linear intervals, we now have to divide the bounding surface into a series of surface elements. These may be rectangular or triangular in shape as in the two-dimensional finite-element formulation. The unknown functions are assumed constant over each element, and the integrals are replaced by sums over all the elements. If the surface is divided into  $m$  elements, we get  $3m$  simultaneous equations to solve.

The stresses are given by

$$\sigma_{ij}(P) = \int_S (V_{ijk}P_k - T_{ijk}u_k) dS + \int_V V_{ijk}F_k dV - 2G\epsilon_{ij}^p + \int_V T_{ijkl}(\epsilon_{kl}^p + \delta_{kl}\alpha T) dV \quad (58)$$

where  $V_{ijk}$ ,  $T_{ijk}$ ,  $T_{ijkl}$  are as given by equations (55) and (57). The general elastoplastic flow theory leading to these equations is given in reference 34.

Using this approach solutions have been obtained for several elastic problems in references 27 and 35 to 37 among others. Figures 7 and 8 show two such problems. Figure 7 (from ref. 35) shows the results for the simple problem of a cube loaded in tension at one end, while on the other end, as well as on two normal faces, the normal displacement components are set equal to zero. The results are shown for 12 surface elements, each surface being divided into two triangles. Good agreement is obtained with the exact solution.

An example of a much more difficult problem involving a semielliptical surface flaw in a plate subjected to tension is shown in figure 8 (from ref. 37). An exact solution is not available here for comparison. A similar problem is treated in reference 30.

## CONCLUDING REMARKS

This brief survey indicates the potential of the boundary integral methods for solving two- and three-dimensional elastic and elastoplastic problems. The more important advantages of these methods are the reduction of the problem size, since the unknown quantities appear only on the boundaries, and the ability to handle arbitrarily shaped

boundaries. This advantage is particularly apparent in using the direct-method involving the Navier equations, since neither smooth boundaries nor simply connected regions are required.

The methods described are not limited to elastostatic or elastoplastic analysis of isotropic materials. References 38 to 40 formulate the general transient elastodynamic problem. Reference 41 discusses the linear viscoelastic problem; references 42 and 43 present solutions to anisotropic elastic problems. Problems with inhomogeneities are discussed in references 27 and 44.

Finally it should be pointed out that, although the theory pertaining to these methods has been explored in some detail, the numerical applications are still in their early stages. Thus, for example, reference 29 discusses improvements that might be made by assuming linear variations of the unknown functions over the boundary intervals rather than assuming them constant over each interval. Reference 13 indicates that taking some intervals to be gradually varying in size, rather than having them all the same size, can improve both the stability and accuracy of the solution. The question of using iterative solutions of the integral equations involved (as was done, for example in ref. 19), instead of reducing them to sets of algebraic equations, has not been fully explored. Thus a great deal of work remains to be done in developing the proper numerical techniques for using these powerful boundary integral methods to their utmost potential.

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501-21.

## APPENDIX - SOME MATHEMATICAL DERIVATIONS

### Derivation of Equation (13)

Substituting  $u = F$  and  $v = \log r$  into equation (12) gives, since  $v = 0$  and  $\nabla^2 F = -2G\alpha = f(Q)$ ,

$$- \iint_R f(Q) \log r_{PQ} \, dx \, dy = \int_C \left[ F(\log r_{PQ})_q - F' \log r_{PQ} \, dq \right] \quad (A1)$$

Now the integrands in equation (A1) are singular when  $r = 0$ . We must therefore exclude this point from the region  $R$  by drawing a small circle of radius  $\epsilon$  about this point and making a cut as shown in figure 9. The line integration is performed as shown, and the limit taken as  $\epsilon$  shrinks to zero. Thus,

$$\begin{aligned} \int_{C+C_\epsilon} F \frac{\partial \log r}{\partial n} \, dq &= \int_C F \frac{\partial \log r}{\partial n} \, dq - \int_{C_\epsilon} F \frac{\partial \log r}{\partial n} \, dq \\ &= \int_C F \frac{\partial \log r}{\partial n} \, dq - \int_0^{2\pi} F \frac{1}{\epsilon} \, d\theta \\ &= \int_C F \frac{\partial \log r}{\partial n} \, dq - 2\pi F(P) \quad \text{as } \epsilon \rightarrow 0 \end{aligned} \quad (A2)$$

and

$$\int_{C+C_\epsilon} \log r \frac{\partial F}{\partial n} \, dq = \int_C \log r \frac{\partial F}{\partial n} \, dq - \epsilon \log \epsilon \int_0^{2\pi} \frac{\partial F}{\partial n} \, d\theta = \int_C \log r \frac{\partial F}{\partial n} \, dq \quad \text{as } \epsilon \rightarrow 0 \quad (A3)$$

which results in equation (13).

For a point on the boundary a semicircle must be used instead of a circle to exclude the singular point leading to a  $\pi F(p)$  term instead of a  $2\pi F(P)$  term and resulting in equation (14).

### Derivation of Equations (31) and (33)

Substituting  $u = \nabla^2 F$  into equation (12) results in

$$\iint_R \nabla^2 F \nabla^2 v \, dx \, dy = \iint_R v \nabla^4 F \, dx \, dy + \int_c \left[ \nabla^2 F \frac{\partial v}{\partial n} - v \frac{\partial(\nabla^2 F)}{\partial n} \right] dq \quad (A4)$$

Since the left side is symmetric in  $F$  and  $v$ , we can also write

$$\iint_R \nabla^2 F \nabla^2 v \, dx \, dy = \iint_R F \nabla^4 v \, dx \, dy + \int_c \left[ \nabla^2 v \frac{\partial F}{\partial n} - F \frac{\partial(\nabla^2 v)}{\partial n} \right] dq \quad (A5)$$

Subtracting the second equation from the first gives

$$\iint_R (F \nabla^4 v - v \nabla^4 F) \, dx \, dy = \int_c \left[ F \frac{\partial(\nabla^2 v)}{\partial n} - \frac{\partial F}{\partial n} \nabla^2 v - v \frac{\partial(\nabla^2 F)}{\partial n} + \frac{\partial v}{\partial n} \nabla^2 F \right] dq \quad (A6)$$

Let  $v = r^2 \log \equiv \rho$  where  $r$  is as previously defined. Then  $\nabla^4 v = 0$ , and, since  $\nabla^4 F = f(x, y)$ ,

$$- \iint_R f(\xi, \eta) \rho(x, y, \xi, \eta) \, d\xi \, d\eta = \int_c \left[ F \frac{\partial(\nabla^2 \rho)}{\partial n} - \frac{\partial F}{\partial n} \nabla^2 \rho - \rho \frac{\partial(\nabla^2 F)}{\partial n} + \frac{\partial \rho}{\partial n} \nabla^2 F \right] dq \quad (A7)$$

As before, the singular point in the region must be excluded before evaluating the line integral. This is done by drawing a small circle of radius  $\epsilon$  about this point, making a cut (as shown in fig. 9), performing the integration shown, and taking the limit as  $\epsilon$  shrinks to zero. Only the first term of the line integral makes a contribution. Thus

$$\lim_{\epsilon \rightarrow 0} \int_{c_\epsilon} F \frac{\partial(\nabla^2 \rho)}{\partial n} \, dq = -\lim_{\epsilon \rightarrow 0} \int_{c_\epsilon} F \frac{4}{\epsilon} \epsilon \, d\theta = -8\pi F(P) \quad (A8)$$

Equation (A7) then becomes

$$8\pi F(\mathbf{x}, \mathbf{y}) = \iint_R \rho f(\xi, \eta) d\xi d\eta + \int_c \left[ F \frac{\partial(\nabla^2 \rho)}{\partial n} - \frac{\partial F}{\partial n} \nabla^2 \rho + \frac{\partial \rho}{\partial n} \nabla^2 F - \rho \frac{\partial(\nabla^2 F)}{\partial n} \right] dq \quad (A9)$$

For a point on the boundary a semicircle is drawn to exclude the singular point resulting in the term  $4\pi F(p)$  rather than  $8\pi F(P)$ .

#### Derivation of Equations (50) and (51)

Betti's reciprocal theorem states: Given two states of stress and displacement due to two different sets of loads, then

$$\int_c P_j u_j^* dq + \int_R F_j u_j^* dA = \int_c P_j^* u_j dq + \int_R F_j^* u_j dA \quad (A10)$$

For the problem under consideration,  $F_j$  and  $P_j$  are taken to be equal to  $F_j'$  and  $P_j'$  (as given by eqs. (47) or (49)), and  $u_j$  is the displacement field for the body. For the second stress state we choose Kelvin's singular solution for a point load in an infinite medium. This solution (refs. 6 and 25) is given by equations (51) where  $U_{ij}$  is the displacement in the  $x_j$  direction due to a unit load in the  $x_i$  direction a distance  $r$  away, and  $T_{ij}$  is the corresponding traction tensor at the same point. Thus, for a unit concentrated force acting in the direction given by the unit base vector  $e_k$ ,

$$\left. \begin{aligned} u_j^* &= U_{kj} e_k \\ P_j^* &= T_{kj} e_k \\ F_j^* &= 0 \end{aligned} \right\} \quad (A11)$$

and

Substituting into equation (A10) gives

$$\int_{c+c_\epsilon} U_{kj} P_j' dq + \int_{R-R_\epsilon} U_{kj} F_j' dA = \int_{c+c_\epsilon} T_{kj} u_j dq \quad (A12)$$

where  $R_\epsilon$  with boundary  $c_\epsilon$  is the circular region excluded because of the singular nature of the Kelvin solution. Taking the limit as  $\epsilon$  goes to zero, in a manner similar to

what was done in deriving equation (13), results in equation (50). The term  $\lambda u_i(P)$  comes from the term on the right hand side of equation (A12).

Equation (52) is obtained as follows. The elastic part of the total strains is given by

$$\epsilon_{ij}^e = \epsilon_{ij} - \epsilon_{ij}^p - \delta_{ij} \alpha T \quad (A13)$$

Now

$$\int_{R-R_\epsilon} \sigma_{ij}^* \epsilon_{ij}^e dA = \int_{R-R_\epsilon} \sigma_{ij}^* \epsilon_{ij}^* dA \quad (A14)$$

where the starred fields refer, as before, to Kelvin's singular solution. Equation (A14) follows from Hooke's law. Substituting equation (A13) into equation (A14) and using the strain displacement relations, the equilibrium equations for the starred stress field, and the divergence theorem result in

$$\int_{c+c_\epsilon} u_i \sigma_{ij}^* n_j dq - \int_{R-R_\epsilon} \sigma_{ij}^* (\epsilon_{ij}^p + \delta_{ij} \alpha T) dA = \int_{c+c_\epsilon} u_i^* \sigma_{ij} n_j dq + \int_{R-R_\epsilon} u_i^* F_i dA \quad (A15)$$

Utilizing the relations

$$\left. \begin{aligned} \sigma_{ij}^* n_j &= P_i^* = T_{ki} e_k \\ u_i^* &= U_{ki} e_k \\ \sigma_{ij}^* &= \Sigma_{ijk} e_k \end{aligned} \right\} \quad (A16)$$

where  $T_{ki}$ ,  $U_{ki}$ , and  $\Sigma_{ijk}$  are defined in equations (51) and (53), gives

$$\int_{c+c_\epsilon} T_{ki} u_i dq - \int_{R-R_\epsilon} \Sigma_{ijk} (\epsilon_{ij}^p + \delta_{ij} \alpha T) dA = \int_{c+c_\epsilon} U_{ki} P_i dq + \int_{R-R_\epsilon} U_{ki} F_i dA \quad (A17)$$

Taking the limit as  $\epsilon$  approaches zero results in equation (52).

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TABLE I. - RESULTS FOR ELLIPSE AND SQUARE<sup>a</sup>

Number of equations, n	Dimensionless torsional rigidity, D	Maximum dimensionless shear stress, $\tau_m$
Ellipse		
2	4.985	2.134
4	5.087	1.645
8	5.033	1.603
Analytical	5.026	1.600
Square		
2	2.189	1.344
4	2.234	1.349
8	2.246	1.350
Analytical	2.249	1.350

<sup>a</sup>From ref. 9.

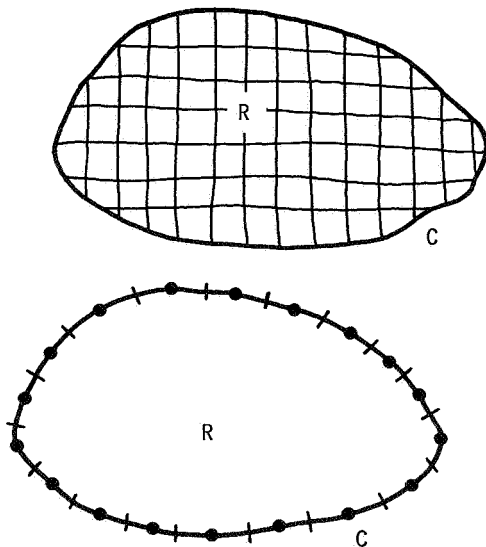


Figure 1. - Interior and boundary modal points.

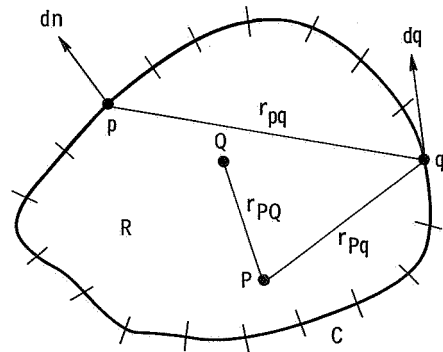


Figure 2. - Region R, boundary curve c, and geometric quantities entering into boundary integrals.

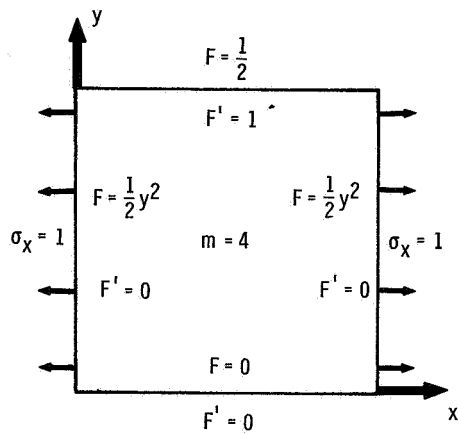


Figure 3. - Square plate under uniform tension. Exact solution:  $\sigma_x = \varphi = 1$ . Boundary-integral solution,  $\sigma_x = 1.000$ .

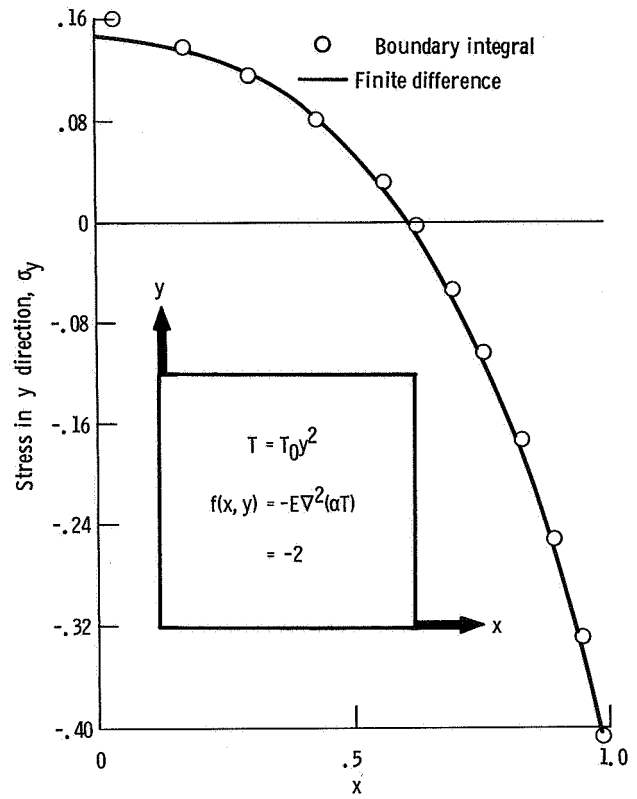


Figure 4. - Thermal stress in square plate;  $\sigma_y$  at  $y = 0$  as function of  $x$ . (One quadrant of plate shown.)

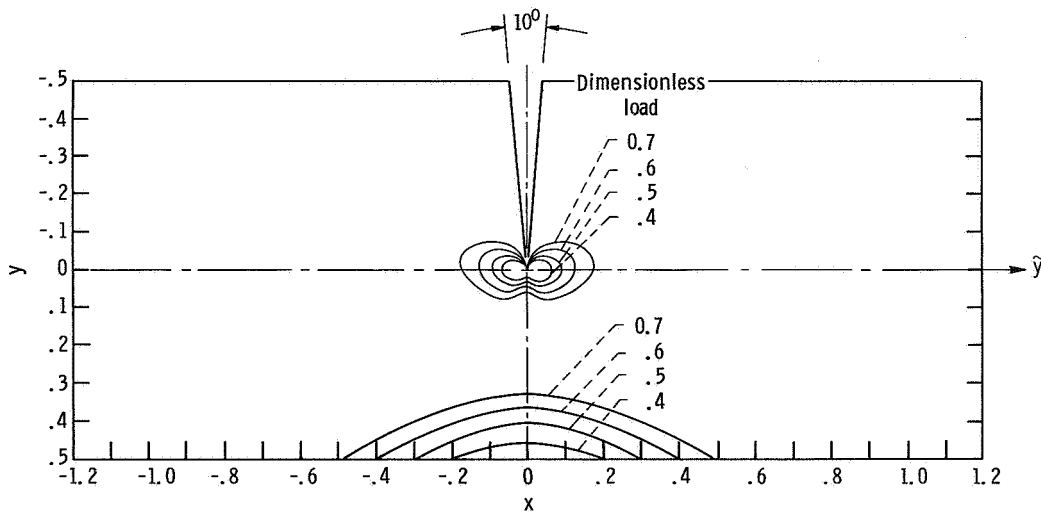
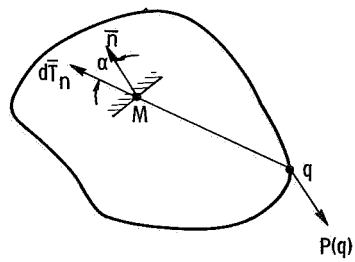
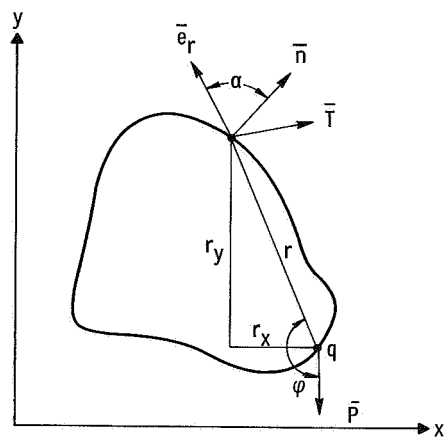


Figure 5. - Growth of the plastic zone size with load for a linear strainhardening specimen with a  $10^0$  edge notch subjected to pure bending; plane strain; ratio of slope of strainhardening line to elastic line, 0.05 (from ref. 13).



(a) Stress vector at interior point M due to fictitious load distribution.



(b) Boundary forces and variables entering into fictitious load formulation.

Figure 6. - Geometric and force variables entering into fictitious load formulation (eqs. (43) and (44)).

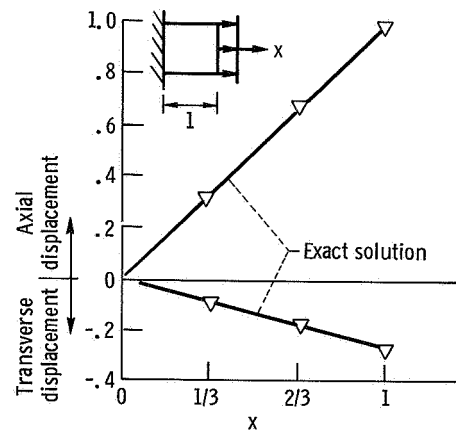
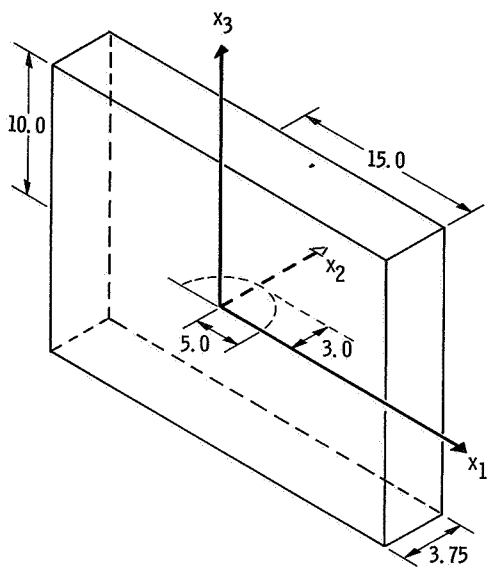
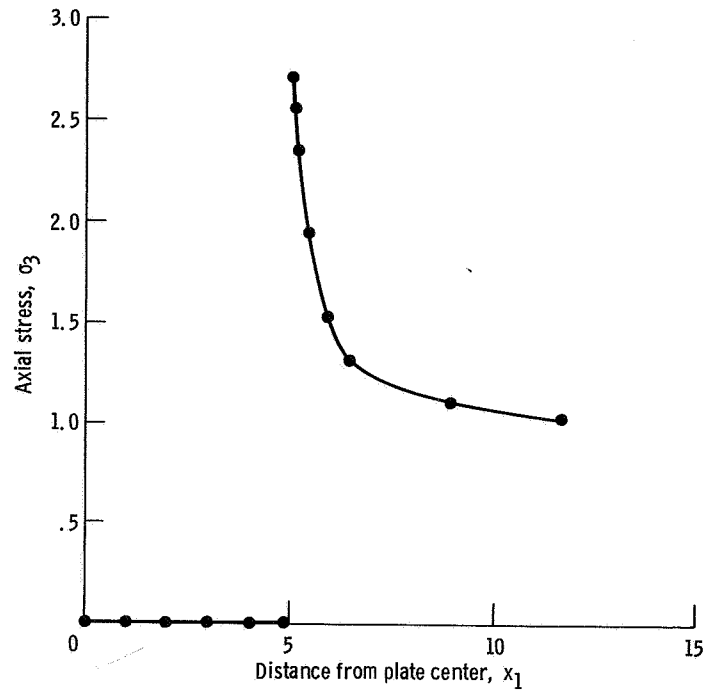


Figure 7. - Unit cube under axial load for 12 surface elements.



(a) Geometry and relative dimensions.



(b) Axial stress distribution along major axis. 256 Surface elements used.

Figure 8. - Semielliptical surface flow in a plate subjected to tension (from ref. 37).

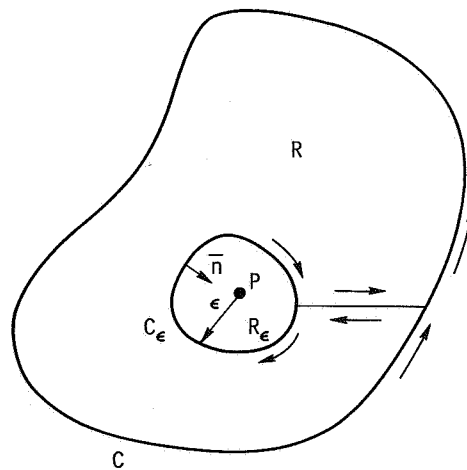


Figure 9. - Integration path excluding singular point.



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